THE BRITTLE FRACTURE OF CEMENTED BODIES

(O KRUPKOM RAZRUSHENII SKLEENNYKH TEL)

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The investigation of a fracture of cemented bodies due to the propagation of a crack in the region of the cemented joint has an important practical significance. This type of fracture refers, for example, to the advancement of a crack originating as a result of a hydraulic discontinuity and propagating along the boundary of the division between two strata of a rock. However, until recently, the brittle fracture of cemented bodies had not been investigated theoretically.

In the theory of cracks of brittle fracture, mainly the problems of cracks in homogeneous solids have been considered. An account of the basic theory of cracks in homogeneous bodies has been given by Barenblatt [1]. The majority of experimental investigations are also concerned with cracks in homogeneous bodies. A survey of these investigations can be found in recent articles by Hiorns and Venables [2,3].

The propagation of cracks in the region of a cemented joint differs qualitatively from the propagation of cracks in homogeneous bodies. Let us stop with this distinction, assuming here and in the future that a plane deformation takes place.

In the homogeneous material the evolution of a crack proceeds in the following manner. As the load increases, which tends to enlarge the crack, the tip of the crack initially remains immobile. Thereby the distribution of the forces of cohesion acting between opposite sides of a crack in a small terminal region, and the form of the crack in this region is changed. This change ceases at the instant the quasistatic (i.e. mobile equilibrium) advancement of the tip of the crack begins. If such an advancement is possible, then it proceeds in a direction insuring local symmetry, i.e. only normal stresses act in the terminal region, symmetrically distributed relative to the direction of propagation. The form of the crack and the distribution of forces of cohesion in the terminal region of the quasistatically advancing terminus of the crack no longer depend on the surface parameters, but are determined only by the properties of the material. The tip region becomes autonomous.

Let us now consider a crack found at the cemented interface between two elastic homogeneous materials. If the strength of the cemented joint is sufficiently great, the crack will not proceed along this surface, but will be propagated deep into one, or simultaneously into two, cemented bodies in conformity with the propagation of a crack in homogeneous materials.

Of fundamental interest is the consideration of a very different kind of case; that is, when the strength of the cemented joint is relatively small. The crack will then be propagated along the surface of the cemented joint and its development will be completely different. In the process of quasistatic advancement, the terminus region of such a crack already has no local symmetry because the terminus of the crack advances along a predetermined course – the border of the cemented joint – which is independent of the manner of applying the load. Due to the differences in the properties of the cemented materials, the opposite sides of the crack may bulge and be superimposed on one another, forming small areas of contact. The reactions acting on these small areas will be added to other acting forces and influence the extension of the crack. It is also characteristic of the case of cracks propagating along the surface of the cemented joint between different materials that the action of a pure tension across the crack, or a pure shear along the crack in the continuous portion of the solid body, will induce both tangential and normal stresses in both cases.

The differences pointed out do not allow for a simple formal transfer of the methods of the theory of cracks in homogeneous materials to the case of cracks advancing along the surface of a cemented joint.

In the present article, with the aid of a suitable extension of the methods of the theory of cracks in homogeneous materials, the condition of equilibrium of a crack is developed which can advance only over the surface of a cemented joint, and the method for determining the advancement of the terminal locations of the crack is shown.

1. The investigation of the field of stresses and displacements in the vicinity of the terminus of a linear cut. Let us assume that the boundary of the cemented joint is rectilinear, coinciding with the x-axis, and the cut extending along it from $x = 0$ to $x = l$ is loaded by distributed stresses over its sides; no other loads are present. The cemented bodies will be assumed infinite; their elastic constants will be designated by the indices 1 and 2 for $y > 0$ and $y < 0$ respectively. For $y > 0$, corresponding to equations by Muskhelishvili [4], we have for the stresses σ_x, σ_y and τ_{xy}, and for the displacements u and v, the following:

$$\sigma_x + \sigma_y = 4 \operatorname{Re} \Phi(z), \qquad \sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z}) \overline{\Phi'(z)}$$
$$2\mu_1 \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) = \varkappa_1 \Phi(z) - \Omega(\bar{z}) - (z - \bar{z}) \overline{\Phi'(z)} \tag{1.1}$$

where μ is the shear modulus, $\kappa = 3 - 4\nu$, ν is Poisson's ratio and $z = x + iy$.

Let us consider the problem which is a particular case of one of the problems investigated by Cherepanov in [5], and independently, but at a later time, by Erdogan [6]. The solution of this problem for $y > 0$ has the form

$$\Phi = \frac{\mu_1(\varkappa_2 + 1) F_1 + (\mu_2 + \mu_1\varkappa_2) F_2}{\mu_1(\varkappa_2 + 1) + \mu_2(\varkappa_1 + 1)}, \qquad \Omega = \frac{-\mu_1(\varkappa_2 + 1) F_1 + (\mu_1 + \mu_2\varkappa_1) F_2}{\mu_1(\varkappa_2 + 1) + \mu_2(\varkappa_1 + 1)}$$
$$F_1 = \frac{1}{2\pi i} \int_0^l \frac{p^+(t) - p^-(t)}{t - z} dt, \qquad F_2 = \frac{1}{2\pi i Z(z)} \left[\int_0^l \frac{f(t) Z(t + i0)}{t - z} dt + iC \right] \tag{1.2}$$
$$Z(z) = z^{1/2 - i\beta} (z - l)^{1/2 + i\beta}, \qquad \lim_{z \to \infty} [Z(z)/z] = 1$$

where

$$m = \frac{\mu_1 + \mu_2\varkappa_1}{\mu_2 + \mu_1\varkappa_2}, \qquad \beta = \frac{1}{2\pi} \ln m, \qquad C = \frac{\mu_2\mu_1(\varkappa_2\varkappa_1 - 1)}{(\mu_1 + \mu_2\varkappa_1)(\mu_2 + \mu_1\varkappa_2)} (X + iY)$$
$$f = \frac{\mu_2(\varkappa_1 + 1) p^+ + \mu_1(\varkappa_2 + 1) p^-}{\mu_2 + \mu_1\varkappa_2}, \qquad p^\pm = (\sigma_y - i\tau_{xy})|_{z = x \pm i0} \tag{1.3}$$

The quantities X and Y represent the components of the resultant forces applied to the surfaces of the cut. The solution for $y < 0$ is derived from the reduced solution with the aid of obvious redesignations. From the written-out solution, in the vicinity of the terminus of the cut $x = 0$, with an accuracy of up to an infinitesimal of higher order, we get the following equation:

$$[u + iv] = M \sqrt{x} \left(\frac{x}{l - x} \right)^{i\beta} \{(B - 2\beta A) - i(A + 2\beta B)\} \tag{1.4}$$

where, on the left, stands the difference of the displacement vectors of the upper and lower sides of the cut at corresponding points, M is some positive constant remaining finite as β approaches 0; the quantities A and B are expressed by the applied stresses

$$A + iB = \frac{1}{2\pi \sqrt{l}} \left\{ \frac{1}{\sqrt{m}} \int_0^l \left(\frac{l-t}{t} \right)^{i\beta + 1/2} f(t)\, dt + C \right\} \qquad (1.5)$$

From (1.4) it follows that the differences of the longitudinal and lateral displacements of the sides of the cut for $A^2 + B^2 \neq 0$ is represented by an oscillating function. For $x \to (l/2)$ the exponent in (1.4) approaches unity. As x decreases from $l/2$ to 0, it becomes purely imaginary for the first time for

$$x = x_* = le^{-\pi/2\beta} (1 + e^{-\pi/2\beta})^{-1} \qquad (1.6)$$

Here and subsequently it is assumed that $\beta > 0$. This can always be achieved by correspondingly numbering the different boundaries of the semi-spaces. It is easy to demonstrate that, from the other side, $\beta \leqslant (\ln \kappa_1)/2\pi$. Since $\nu \geqslant 0$ is always true, then for all possible values of β the quantity $x_* < 10^{-4} l$. This inequality permits the use of equation (1.5) in the interval of the variable x including x_*.

In the interval $0 < x < x_*$ the vector $[u + iv]$ for $A^2 + B^2 \neq 0$ goes through an infinite number of revolutions. Such an oscillating character of the solution in the vicinity where the shift of the boundary condition takes place at the border of the cut between different bodies was noted long ago by Abramov [7].

From the solutions of (1.1) to (1.3) it also follows that in a continuous body on the extension of the cut near its terminus, $x = 0$

$$\sigma_y - i\tau_{xy} = -\frac{1}{\sqrt{s}} \left\{ \left(\frac{s}{(l+s)} \right)^{i\beta} (A + iB) + o(1) \right\} \qquad (1.7)$$

Here $z = x = -s$, $s \to +0$. From (1.7) it follows that for $A^2 + B^2 \neq 0$ the stresses at the extension of the cut are infinitely large and change the sign an infinite number of times.

2. Derivation of the equilibrium condition for the tip of a crack. By considering the elastic field in the immediate vicinity of the tip of a curvilinear crack, the boundary of the cut can be considered to be linear, and the bodies separated by it are infinite. From this it follows that the distribution of displacements of the sides of the crack and the stresses on the extension of the crack in the small terminal region will be the same as in the immediate vicinity of the terminus of a linear semi-infinite cut. In the formulas (1.4) and (1.7), transferring to asymptotic expressions for $l \to \infty$, we find

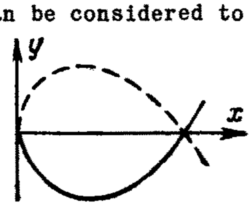

Fig. 1.

that in this vicinity we obtain

$$[u + iv] = M\sqrt{x}\left(\frac{x}{L}\right)^{i\beta}\{(B - 2\beta A) - i(A + 2\beta B)\}$$

$$\sigma_y - i\tau_{xy} = -\frac{1}{\sqrt{s}}\left(\frac{s}{L}\right)^{i\beta}(A + iB) \tag{2.1}$$

Here the x-axis is directed along the tangent to the length of the crack, the y-axis is normal to it, the length L is related to the geometrical characteristics of the problem, the values A and B depend upon the loads acting on it (in the general case, not according to formula (1.5)).

In Section 1 it was noted that, approaching the tip of the crack, the vector, equal to the difference of the displacement vectors of the upper and lower sides, goes through an infinite number of revolutions. This indicates that, in an infinite number of points, the upper side appears to be under the lower one. Such a physically impossible case of interpenetration of regions divided by a crack was also previously noted for cracks in homogeneous materials [8]. By analyzing the structure of the crack near its tip, it was shown that this structure has the form depicted in Fig. 1, where the upper side is located underneath the lower one. The stresses at the extension of such a "crack" appear to be in compression and they hinder its extension into the depth of the body. On the contrary, they bar its joining until the moment the small area of contact from $x = 0$ to $x = x_1$ is formed. The separation x_1 is determined from the condition of smoothness of the displacement of opposite sides of the crack, or by the finiteness of the stresses. This condition, first introduced by Khristianovich [9], and later proved by Barenblatt [10], using the calculus of variations, is fundamental to the theory of cracks.

In the case under discussion, for $A^2 + B^2 \neq 0$ in the vicinity of the tip of a crack, there are an infinite number of places in which it would be necessary for interpenetration of the mediums to occur. For this reason, generally speaking, several small areas of contact may be formed so that the crack in the vicinity of the terminus will have the appearance depicted in Fig. 2. The reaction forces will act on these small areas. Further, the forces of cohesion will act in the terminus region of the crack. As a result, the values of A and B will be combined from the values of A_0 and B_0, calculated without considering the reactive forces and forces of cohesion, and the quantities A' and B', taking into account the action of these forces

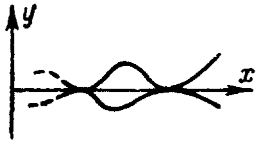

Fig. 2.

$$A = A_0 + A', \qquad B = B_0 + B' \tag{2.2}$$

In the theory of cracks of homogeneous materials, the condition of mobile equilibrium of a given point on the contour of a crack is derived by finding the virtual work and setting it equal to zero. This was done in [10,11].

We will proceed analogously in the case under consideration.

Let us separate the elastic field into two parts: the field which originates from the applied loads in the absence of cracks, and the field created by the crack with the load in the form of stresses, superimposed on those which the first field produced at the location of the crack. For a variation of the location of the terminus of the crack the first field will not change and, consequently, does not appear in the expression for work. This expression will completely determine the second field. Thus, for the calculation of virtual work, we may consider that the entire load is applied to the upper crack and that there are no other loads. The virtual displacement of the tip of the crack along the boundary of the cemented joint δh will be assumed so small that in the investigation of the region affected by variations, this boundary can be considered to be rectilinear, and the mediums divided by it to be infinite. In this way the problem leads to the calculation of virtual work, which is produced by a virtual displacement of a semi-infinite cut along the rectilinear boundary between two semi-spaces.

Each point of the cut, as a result of a virtual displacement, is shifted a distance δh. Thus, for example, the force $p^{+}(x)\delta h$ acts on the upper side at the point x. The displacement at this point is changed to $[\partial(u^{+} + iv^{+})/\partial x]\,dx$. As a result, the summation of work δw^{+} is carried out over the upper side

$$\delta w^{+} = -\,\delta h \int\limits_{0}^{\infty} \left(\frac{\partial u^{+}}{\partial x}\,\tau_{xy}{}^{+} + \frac{\partial v^{+}}{\partial x}\,\sigma_{y}{}^{+}\right) dx = \,-\,\delta h\,\mathrm{Im}\int\limits_{0}^{\infty} \overline{p_{+}} \left(\frac{\partial u^{+}}{\partial x} + i\,\frac{\partial v^{+}}{\partial x}\right) dx \quad (2.3)$$

The minus sign is placed in front of the integral because, on the upper side, the positive direction of the stresses $\tau_{xy}{}^{+}$, $\sigma_{y}{}^{+}$ is opposite to the positive directions of the x- and y-axes, respectively. In this expression it is necessary to substitute the asymptotic solution, obtained from (1.1) to (1.3) by passing to the limit $l \to \infty$. As a result, we obtain

$$\delta w^{+} = -\,\mathrm{Im}\,\frac{\varkappa_1 + 1}{4\pi i\mu_1\,[\mu_2\,(\varkappa_1 + 1) + \mu_1\,(\varkappa_2 + 1)]}\left\{\mu_2\,(\varkappa_1 + 1)\iint\limits_{0}^{\infty} \frac{p^{+}(t)\,\overline{p^{+}(x)}}{t - x}\left(\frac{t}{x}\right)^{1/2 - i\beta} dt\,dx \,+\right.$$

$$\left. +\,\mu_1\,(\varkappa_2 + 1)\left[\iint\limits_{0}^{\infty} \frac{p^{-}(t)\,\overline{p^{+}(x)}}{t - x}\left(\frac{t}{x}\right)^{1/2 - i\beta} dt\,dx - \iint\limits_{0}^{\infty} \frac{p^{-}(t)\,\overline{p^{+}(x)}}{t - x}\,dt\,dx\right]\right\}\delta h \quad (2.4)$$

Designating the first integral in (2.4) by J_1, we convert it into the following form:

$$J_1 = \iint_0^\infty \frac{p^+(t)}{\sqrt{t}} \frac{\overline{p^+(x)}}{\sqrt{x}} \frac{t-x+x}{t-x} \left(\frac{t}{x}\right)^{-i\beta} dt\,dx = \iint_0^\infty \frac{p^+(t)}{\sqrt{t}} \frac{\overline{p^+(x)}}{\sqrt{x}} \left(\frac{t}{x}\right)^{-i\beta} dt\,dx - \bar{J}_1 \tag{2.5}$$

Similarly, for the second interval, we find

$$J_2 = \iint_0^\infty \frac{p^-(t)}{\sqrt{t}} \frac{\overline{p^+(x)}}{\sqrt{x}} \left(\frac{t}{x}\right)^{-i\beta} dtdx - \iint_0^\infty \frac{p^-(x)\,\overline{p^+(t)}}{t-x} \left(\frac{t}{x}\right)^{i\beta} dtdx \tag{2.6}$$

The expression for the work on the lower side δw^- is obtained from (2.4) by substituting κ_1 for κ_2, μ_1 for μ_2, p^+ for $\overline{p^-}$ and p^- for $\overline{p^+}$. Combining the expressions for δw^+ and δw^-, we find, taking into account (2.5) and (2.6), that the total work δw is equal to

$$\delta w = \frac{(\mu_2 + \mu_1\varkappa_2)^2}{8\pi\,\mu_1\mu_2\,[\mu_2(\varkappa_1+1) + \mu_1(\varkappa_2+1)]} \left| \int_0^\infty \frac{f(t)}{\sqrt{t}} \left(\frac{t}{L_0}\right)^{-i\beta} \right| \delta h \tag{2.7}$$

where L_0 is an arbitrary quantity with the dimension of a length. The work δw is independent of this quantity. This is immediately obvious if the expression for the modulus is given explicitly.

Assuming in (1.5) $l = L_0$ and changing to the asymptotic expression for $L_0 \to \infty$, after substituting this expression into (2.7), we obtain

$$\delta w = \frac{\pi(\mu_1 + \mu_2\varkappa_1)(\mu_2 + \mu_1\varkappa_2)}{2\mu_1\mu_2\,[\mu_2(\varkappa_1+1) + \mu_1(\varkappa_2+1)]} (A^2 + B^2)\,\delta h \tag{2.8}$$

The condition of equilibrium for the given tip of the crack is contained in the equation $\delta w = 0$. Assuming in (2.8) that $\delta w = 0$, and using the notation given in (2.2), we obtain

$$A \equiv A_0 + A' = 0, \qquad B \equiv B_0 + B' = 0 \tag{2.9}$$

From these equalities and formulas (2.1), it follows that if the tip of the crack under consideration is in equilibrium, then in the small terminal region the opposite sides of the crack, as in the case of a crack in a homogeneous material, are smoothly and evenly displaced. As a result of this, the stresses at the continuation of the crack become finite. In this way, taking account of the possible occurrence of small areas of contact, the structure of a crack in the vicinity of its mobile equilibrium tip has the form depicted in Fig. 3.

3. The conditions for the determination of the location of the tip of a crack. The foundation of the current theory of cracks is stated in two hypotheses [12]: the hypothesis of the smallness of the tip region of the crack compared to the overall macroscopic dimensions of the problem; and the hypothesis of the autonomy of this terminal region. Autonomy is understood in the sense in which it was explained in the introduction.

In the case under consideration, the terminal region is the one where the forces of interaction (the forces of cohesion and the forces of reaction) of the opposite sides of the crack are distributed. If the external loads are distributed so that, by calculating the displacements using (1.1) to (1.3), the locations where the interpenetration of the media would occur, they are concentrated only in the vicinity of the crack, while in the main section of the crack such places are not found, then the small areas of contact may be found only in the vicinity of the tip of the crack. The size of the region where these small areas are distributed approaches zero with the disappearance of the difference in the properties of the media, between which the crack is located. In the case of a rectilinear crack of length l in an infinite body, the size of each of these regions, as was shown, never exceeds $10^{-4}l$. Thus, disregarding the fact that the region of action of the forces of cohesion, on account of the specific nature of these forces, is always small, we may consider that the terminal region is also small, i.e. the first hypothesis is valid.

The first hypothesis. The terminal region, where the forces of interaction of opposite sides of the crack are distributed, is small in comparison to the characteristic dimensions.

This hypothesis can be used at least until the size of the crack is of an order of magnitude less than, or equal to, the remaining characteristic dimensions, which are independent of it.

It is impossible, in the case under consideration, to accept the second hypothesis (which deals with the autonomy of a terminal region) in the same form in which it was formulated for cracks in homogeneous materials, because, with the movement of the crack over the surface of the cemented joint in the terminal regions, there is no local symmetry.

The location of the given tip of the crack is determined by the elastic field produced by the forces of interaction distributed in the terminal region. This field, in its turn, is uniquely determined by the quantities A' and B'. Let us assume that the applied load is changed, depending on some parameter λ in such a way that the given tip of a crack quasistatically extends deep into the body. To each value of the parameter λ there is a corresponding unique location of this tip and unique values of the quantities A' and B'. Hence, with this load distribution

$$F(A', B') = 0 \tag{3.1}$$

The form of the function, defined by (3.1), can be varied depending mainly upon how the applied loads are changed. This function in general may not exist if the variations of the loads depend not on one but, for example, on two parameters. The assumption that for mobile-equilibrium (quasistatic movement) of the tip of the crack, the function (3.1) exists, and its form does not depend upon the applied forces, is the natural generalization for accepting autonomy in the case being considered. In the same way, the second hypothesis can be accepted in the generalized form given below.

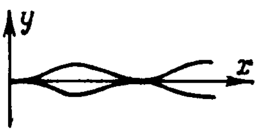

Fig. 3.

The second hypothesis (the generalized hypothesis of autonomy). If the tip of the crack that can advance only over the surface of the cemented joint between two elastic materials if found to be in a state of mobile equilibrium, then

$$F(A', B') = 0 \tag{3.2}$$

where the function, defined by (3.2), exists and does not depend upon the external loads. This function is determined by the properties of the cemented materials, the cement itself and their thermodynamic state.

We will show that, in the case when materials between which the crack is propagated are different, the form of this function is determined in a unique way. From (2.1), we have

$$A' + iB' = -\lim_{s\to 0} (\sigma_y' - i\tau_{xy}') \sqrt{s} \left(\frac{s}{L}\right)^{-i\beta} \tag{3.3}$$

where σ_y' and τ_{xy}' are the stresses at the continuation of the crack, caused by the forces of interaction. In formula (3.3) all the quantities, except L depend on the properties of the elastic field in the vicinity of the tip of the crack. The geometrical parameter L, in general, depends on the applied loads. If the generalized hypothesis of autonomy is assumed, then the left-hand side of (3.2) clearly must not depend on L. It is easy to see with the aid of (3.3) that the only combinations of A' and B' which clearly do not depend upon L are $A'^2 + B'^2$ and any other function of this quantity. As a result, condition (3.2) obviously must have the form

$$A'^2 + B'^2 = D^2 \tag{3.4}$$

where D is a constant quantity, depending on the properties of the cement, of the cemented materials, its thermodynamic state, etc., and is not dependent on the applied loads. Combining (3.4) and (2.8), we see that the assumption of autonomy leads to the necessity for the constancy of work, produced only by the forces of interaction, associated with the formation of a unit length of the crack.

The position of the mobile-equilibrium terminals of the crack is determined from the condition of autonomy (3.4) and the conditions of equilibrium (2.9). The problem concerning the evolution of the crack from the initial cut for the case under discussion is formulated in the same manner as in the theory of cracks in homogeneous materials.

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